An initial-boundary-value problem that approximate the quarter-plane problem for the Korteweg-de Vries equation.

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Résumé - Nous continuons l'étude du problème de Cauchy et aux limites pour l'équation de Korteweg-de Vries initiée dans [9]. On obtient des effets régularisants globaux uniformes par rapport à la longueur de l'intervalle et nous montrons que la solution du problème aux limites converge, lorsque la longueur de l'intervalle tend vers l'infini, vers la solution du problème posé sur le quart de plan $t > 0$, $x > 0$. Nous proposons un schéma aux différences finies très simple pour le problème sur $[0, 1]$ et montrons sa stabilité.

Abstract - In this paper, we continue the study of the initial-boundary-value problem for the Korteweg-de Vries equation that has been initiated in [9]. We obtain global smoothing effects that are uniform with respect to the size of the interval. This allows us to show that the solution of the boundary value problem converges, as the size of the interval converges to infinity, towards the solution of the quarter-plane problem. We also propose a simple finite differents scheme for the problem on $[0, 1]$ and prove its stability.

Keywords: Korteweg-de Vries, boundary conditions, smoothing effects, quarter-plane.

1 Introduction and statements of the results

1.1 Introduction

The Korteweg-de Vries equation has been introduced in [12] in order to describe the propagation of long water waves in a channel. If $u(x, t)$ denotes the elevation of the free surface of the flow with respect to the equilibrium position at time t and at the position x, this function satisfies

$$u_t + u_x + uu_x + u_{xxx} = 0, \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

The pure Cauchy problem on the whole line for this equation has been extensively studied, see for example [13], [2] for regular initial data. Recent results shows that the Cauchy problem is well-posed in L^2 [7] or even in negative order Sobolev spaces [11]. In laboratory experiments, the wave is created by a "wave maker" at one of the extremities of the channel. In order to describe this situation, J. L. Bona and R. Winther have considered the Korteweg-de Vries equation in a quarter-plane [3], [4]:

$$\left\{\begin{array}{c} u_t + u_x + uu_x + u_{xxx} = 0,\ x \geq 0,\ t \geq 0, \\ u(0,t) = g(t),\ t \geq 0, \\ u(x,0) = u_0(x),\ x \geq 0. \end{array}\right. \tag{1}$$

One of the results that is obtained reads as follows:

Theorem 1 (J.L. Bona, R. Winther) *Suppose u_0 in $H^4(\mathbb{R}^+)$ and g in $H^2_{loc}(\mathbb{R}^+)$ satisfy the compatibility conditions $u_0(0) = g(0)$, $g'(0) + (\partial_x u_0 + u_0 \partial_x u_0 + \partial_x^3 u_0)(0) = 0$, then there exists a unique solution u_∞ in $L^\infty_{loc}(\mathbb{R}^+, H^4(\mathbb{R}^+))$ of (1).*

In [8], [9], [10], an initial-boundary-value problem for the Korteweg-de Vries equation is studied:

$$\left\{\begin{array}{c} u_t + u_x + uu_x + u_{xxx} = 0,\ x \in [0,L],\ t \in [0,T[, \\ u(0,t) = g(t),\ t \in [0,T[, \\ u_x(L,t) = 0,\ t \in [0,T[, \\ u_{xx}(L,t) = 0,\ t \in [0,T[, \\ u(x,0) = u_0(x),\ x \in [0,L], \end{array}\right. \tag{2}$$

where $L > 0,\ T \in]0,+\infty[$ and the following local existence theorem is established:

Theorem 2 (T. Colin, J.M. Ghidaglia) *Let u_0 be in $H^1(0,L)$ and g be in $\mathcal{C}^1(\mathbb{R}^+)$ satisfy the compatibility condition $u_0(0) = g(0)$. Then there exists $T_L > 0$ and a function u^L in $L^\infty(0,T_L;H^1(0,L)) \cap \mathcal{C}([0,T_L];L^2(0,L))$ which solves (2). Moreover, if $|u_0|_{H^1}$ and $|g|_{H^1(\mathbb{R}^+)}$ are small enough, then $T_L = +\infty$.*

This result is proved by energy methods using a long-wave type regularisation introduced in [3]. In [9], some parabolic type smoothing effects are also proved for (2) and this leads to a local existence theorem in L^2 for the nonlinear problem. These smoothing effects are obtain by multiplying the equation by xu and none of the results given in [9] are uniform with respect to the size of the interval. Note that these smoothing effects were also discovered for the quarter-plane by Bona and Winther [5].

The aim of this paper is to obtain a similar result but uniformly with respect to L ; that is existence of u^L on $[0,T_L[$ for u_0^L in $L^2([0,L])$ with $T_L \to +\infty$ and $u^L \to u_\infty$ when L tends towards $+\infty$ provided that $\int_0^L (1+x^2)(u_0^L)^2(x)dx$ is bounded independently of L and where u_∞ is the solution to the quarter-plane problem.

This paper is organized as follows. We first prove the existence of a time T_{min} depending only on $\mid g \mid_{H^1}$ and $\mid u_0 \mid_{L^2((1+x^2)dx)}$ but not on L such that u_L exists on $[0,T_{min}]$ thanks to uniform (with respect to L) smoothing effects (section 2). This construction will also give uniform bounds on u_L that will allow us to perform the limit $L \to +\infty$. We show that the existence time T_L tends to $+\infty$ as L tends towards $+\infty$ (see section 3). A by-product of the proof is that the quarter-plane problem is globally well-posed in the space $L^2((1+x^2)dx)$. In section 4, we present a finite different scheme for the initial-boundary-value problem (2) ; we prove its stability and present some numerical experiments.

Our main result reads as follows (Theorem 5):

Theorem *Consider a family of initial values $u_0^L \in L^2([0,L])$ such that*

$$\sup_{L\geq 1}\int_0^L |u_0^L|^2(x)(1+x^2)dx < +\infty$$

and such that u_0^L tends towards u_0 in $L^2_{loc}(\mathbb{R}^+)$ strongly.
Then, for all $T>0$, if L is large enough, $u^L(x,t)$ the solution of (2) with initial value u_0^L is defined on $[0,T]$ and u^L tends towards u in $L^p(0,T;L^2_{loc}(\mathbb{R}^+))$ strongly for all $1\leq p<+\infty$, where $u(x,t)$ is a solution of (1) with initial data u_0.

1.2 Notations and basic assumptions

Let X be a Banach space, $1\leq p\leq +\infty$ and $-\infty \leq a < b \leq +\infty$. The notation $L^p(a,b;X)$ denotes the Banach space of measurable functions $u:(a,b)\to X$ whose norms are p th-power integrable (essentialy bounded if $p=+\infty$). There are endowed with the norm : $\mid u \mid^p_{L^p(a,b;X)} = \int_a^b \mid u(.,t) \mid_X^p \, dt$ if $p<+\infty$ and $\mid u \mid_{L^\infty(a,b;X)} = \sup_{t\in(a,b)} \mid u(.,t) \mid_X$ if $p=+\infty$.

In this paper, we assume that for $0<L<+\infty$ the initial data u_0 belongs to $L^2(0,L)$ and that xu_0 belongs to $L^2(0,L)$ and we introduce $\mid . \mid$ which denotes the $L^2((1+x^2)dx)$ norm :

$$\mid u_0 \mid = \mid u_0 \mid_{L^2((1+x^2)dx)} = \sqrt{\int_0^L (1+x^2)u_0^2(x)dx}.$$

We introduce the space :

$$E := \{f\in L^1(0,T;L^2((1+x^2)dx)),\ \sqrt{t}f \in L^2(0,T;L^2((1+x^2)dx))\}$$

endowed with the norm :

$$\mid f \mid_E = \int_0^T \sqrt{\int_0^L (1+x^2)f^2(x,t)dx}dt + \sqrt{\int_0^T\int_0^L t(1+x^2)f^2(x,t)dxdt}.$$

We recall that the following well-known inequalities :
* $\mid w \mid^2_{L^2((1+x)dx)} \leq 3 \mid w \mid^2_{L^2((1+x^2)dx)}$.
* $\mid g \mid_{L^\infty(0,T)} \leq (C+\sqrt{T}) \mid g \mid_{H^1(0,T)}$.
* $\mid G \mid_{L^1(0,T;L^2((1+x^2)dx))} \leq \sqrt{T} \mid G \mid_{L^2(0,T;L^2((1+x^2)dx))}$.
* if w in $H^1(0,L)$, with $w(0)=0$ then $\mid w \mid^2_{L^\infty(dx)} \leq 2 \mid w \mid_{L^2(dx)} \mid w_x \mid_{L^2(dx)}$.

2 Existence and uniqueness for the non homogeneous case ($g \neq 0$)

2.1 Local existence and uniqueness on $[0, L]$, $L < +\infty$

Let $T > 0$. We introduce the space :

$$X_T := \{w \in \mathcal{C}([0,T]; L^2((1+x^2)dx)),\ w_x \in L^2(0,T; L^2((1+x)dx)),$$
$$\sqrt{t}w_x \in L^\infty(0,T; L^2((1+x)dx)),\ \sqrt{t}w_{xx} \in L^2(0,T; L^2)\}.$$

This space is endowed with the norm :

$$\mid w \mid_X := \mid w \mid_{L^\infty(0,T;L^2((1+x^2)dx))} + \mid w_x \mid_{L^2(0,T;L^2((1+x)dx))}$$
$$+ \mid \sqrt{t}w_x \mid_{L^\infty(0,T;L^2((1+x)dx))} + \mid \sqrt{t}w_{xx} \mid_{L^2(0,T;L^2)} .$$

For u_0 in $L^2(0, L)$, the unique solution of the linear homogeneous system

$$\left\{ \begin{array}{c} u_t + u_x + u_{xxx} = 0,\ x \in [0, L],\ t \in [0, T[, \\ u(0,t) = 0,\ t \in [0, T[, \\ u_x(L,t) = 0,\ t \in [0, T[, \\ u_{xx}(L,t) = 0,\ t \in [0, T[, \\ u(x,0) = u_0(x),\ x \in [0, L], \end{array} \right. \tag{3}$$

is denoted $S(t)u_0$ (see [9], [10] for the construction of this semi-group). Let ϕ be a smooth function defined over $\mathbb{R}^+$ such that $\phi(0) = 1$ and $\phi(x) = 0,\ \forall x \geq 1$.

The definition of a weak solution for the system (2) is the following :

Definition 1 *A weak solution to (2) on $[0, T]$ is a function $u(x,t) \in X_T$ such that*

$$\tilde{u}(x,t) = u(x,t) - \phi(x)g(t)$$

satisfy

$$\tilde{u}(x,t) = S(t)\tilde{u}_0(x) -$$
$$\int_0^t S(t-s)\{(\tilde{u}(x) + \phi(x)g(s))(\tilde{u}_x(x,s) + \phi'(x)g(s))$$
$$+g(s)(\phi'(x) + \phi'''(x)) + \phi(x)g'(s)\}ds,$$

where $\tilde{u}_0(x) = u_0(x) - \phi(x)g(0)$.

The existence Theorem on $[0, T_{min}]$, T_{min} being independent of L, depending only on $\mid g \mid_{H^1}$ and $\mid u_0 \mid$ reads as follows :

Theorem 3 *Let u_0 be in $L^2((1+x^2)dx)$, g be in $H^1_{loc}(\mathbb{R}^+)$ and $0 < L < +\infty$. Then there exists a unique weak maximal solution defined over $[0, T_L[$ to (2). Moreover, there exists $T_{min} > 0$ independent of L, depending only on $\mid u_0 \mid$ and $\mid g \mid_{H^1}$ such that $T_L \geq T_{min}$. The solution u depends continuously on u_0 and g in the following sense : let a sequence u_0^n converging towards u_0 in*

$L^2((1+x^2)dx)$, let a sequence g^n converging towards g in $H^1_{loc}(\mathbb{R}^+)$ and denote by u^n the solution with data (u_0^n, g^n) and T_L^n its existence time. Then

$$\liminf_{n\to+\infty} T_L^n \geq T_L$$

and for all $T < T_L$, u^n exists on the interval $[0,T]$ if n is large enough and u^n tends towards u in X_T.

In order to prove this result, we first state some smoothing effects for (3), i.e. in the linear case (cf section 2.1.1). Then, in section 2.1.3, we use these estimates in the non linear case in a fixed point procedure.

2.1.1 Uniform estimates on the solutions to the linear homogeneous problem

In this section we consider the linear problem (3). The following proposition gives more precise results that those of [9]:

Proposition 1 *Let u_0 be in $L^2(0,L)$. There exists a continuous function $t \mapsto c(t)$ such that*

$$\mid u \mid_{L^\infty(0,T;L^2((1+x^2)dx))} \leq c(T) \mid u_0 \mid, \tag{4}$$

$$\mid u_x \mid_{L^2(0,T;L^2((1+x)dx))} \leq c(T) \mid u_0 \mid, \tag{5}$$

$$\mid u_x(0,t) \mid_{L^2(0,T)} \leq c(T) \mid u_0 \mid, \tag{6}$$

$$\mid \sqrt{t} u_x \mid_{L^\infty(0,T;L^2)} \leq c(T) \mid u_0 \mid, \tag{7}$$

$$\mid \sqrt{t} u_x(0,t) \mid_{L^2(0,T)} + \mid \sqrt{t} u_{xx}(0,t) \mid_{L^2(0,T)} \leq c(T) \mid u_0 \mid, \tag{8}$$

$$\mid \sqrt{t} u_x \mid_{L^\infty(0,T;L^2((1+x)dx))} \leq c(T) \mid u_0 \mid, \tag{9}$$

$$\mid \sqrt{t} u_{xx} \mid_{L^2(0,T;L^2)} \leq c(T) \mid u_0 \mid . \tag{10}$$

Proof : multiplying (3) by u, xu, x^2u, one gets the estimates (4), (5), (6). Let us recall how one obtains (5) which is the more surprising estimate since it gives a smoothing effect. Multiplying (3) by u and integrating on $[0,L]$ give

$$\frac{d}{dt}\int_0^L |u|^2 dx + u^2(L,t) + u_x^2(0,t) = 0. \tag{11}$$

Multiplying (3) by xu and integrating on $[0,L]$ yield

$$\frac{1}{2}\frac{d}{dt}\int_0^L x|u|^2 dx + \int_0^L x u_x u dx + \int_0^L x u_{xxx} u dx = 0.$$

Integrating by parts gives, using the boundary conditions:

$$\frac{d}{dt}\int_0^L x|u|^2 dx - \int_0^L |u|^2 dx + 3\int_0^L u_x^2 + Lu^2(L,t) = 0,$$

which implies (5) thanks to (11). Taking the L^2 inner product of (3) with u_{xx}, one obtains

$$\frac{d}{dt}\int_0^L u_x^2(x,t)dx + u_x^2(0,t) + u_{xx}^2(0,t) = 0. \tag{12}$$

Multiplying (12) by s and integrating the resulting expression in the temporal variable over $(0,t)$, one obtains after integration by parts and by using (5) the inequalities (7) and (8). Multiply (3) by xu_{xx} leads to

$$\frac{d}{dt}\int_0^L xu_x^2(x,t)dx + 2u_x(0,t)u_{xx}(0,t) - \int_0^L u_x^2(x,t)dx + 3\int_0^L u_{xx}^2(x,t)dx = 0. \tag{13}$$

Taking the product of (13) by s, integrating the resulting expression in the temporal variable over $(0,t)$, integrating by parts and using Cauchy-Schwarz inequality give :

$$t\int_0^L xu_x^2(x,t)dx + 3\int_0^t s\int_0^L u_{xx}^2(x,s)dxds \le \int_0^t\int_0^L xu_x^2(x,t)dxds$$

$$+\int_0^t s\int_0^L u_x^2(x,s)dxds + 2\sqrt{\int_0^t su_x^2(0,s)ds}\sqrt{\int_0^t su_{xx}^2(0,s)ds}.$$

The two first terms on the right-hand side of the above inequality are bounded thanks to the inequality (5) and the last term thanks to the inequality (8). This gives inequalities (9) and (10). ■

2.1.2 Non-homogeneous linear estimates

Thanks to standard duality technics, we obtain estimates that are independent of L for the non homogeneous linear system

$$\begin{cases} v_t + v_x + v_{xxx} = f(x,t), \ x\in[0,L], \ t\in[0,T[, \\ v(0,t) = 0, \ t\in[0,T[, \\ v_x(L,t) = 0, \ t\in[0,T[, \\ v_{xx}(L,t) = 0, \ t\in[0,T[, \\ v(x,0) = 0, \ x\in[0,L]. \end{cases} \tag{14}$$

Proposition 2 *There exists a continuous function $t\mapsto c(t)$ such that if f belongs to E*

$$|\, v\,|_{L^\infty(0,T;L^2((1+x^2)dx))} \le c(T)\,|\, f\,|_{L^1(0,T;L^2((1+x^2)dx))}, \tag{15}$$

$$|\, v_x\,|_{L^2(0,T;L^2((1+x)dx))} \le c(T)\,|\, f\,|_{L^1(0,T;L^2((1+x^2)dx))}, \tag{16}$$

$$|\, v_x(0,t)\,|_{L^2(0,T)} \le c(T)\,|\, f\,|_{L^1(0,T;L^2((1+x^2)dx))}, \tag{17}$$

$$|\, \sqrt{t}v_x\,|_{L^\infty(0,T;L^2(1+x))} \le c(T)\,|\, f\,|_E, \tag{18}$$

$$|\, \sqrt{t}v_x(0,t)\,|_{L^2(0,T)} \le c(T)\,|\, f\,|_E, \tag{19}$$

$$|\, \sqrt{t}v_{xx}(0,t)\,|_{L^2(0,T)} \le c(T)\,|\, f\,|_E, \tag{20}$$

$$| \sqrt{t} v_{xx} |_{L^2(0,T;L^2(dx))} \leq c(T) \mid f \mid_E, \tag{21}$$

$$| v_x |_{L^\infty(0,T;L^2((1+x)dx))} \leq c(T) \mid f \mid_{L^2(0,T;L^2((1+x^2)dx))}, \tag{22}$$

$$| v_x(0,t) |_{L^2(0,T)} + \mid v_{xx}(0,t) \mid_{L^2(0,T)} \leq c(T) \mid f \mid_{L^2(0,T;L^2((1+x^2)dx))}, \tag{23}$$

$$| v_{xx} |_{L^2(0,T;L^2(dx))} \leq c(T) \mid f \mid_{L^2(0,T;L^2((1+x^2)dx))} . \tag{24}$$

Proof : estimates (15), (16), (17) are dual of (4) and (5) since one has

$$v(x,t) = \int_0^t S(t-s) f(x,s) ds.$$

The inequality (15) is an obvious consequence of inequality (4). In order to prove the estimates (16) and (17), we introduce $\psi(t)$ in $L^2(0,T)$ and compute:

$$\left| \int_0^T \mid v_x \mid_{L^2((1+x)dx)} \psi(t) dt \right| \leq \int_0^T \int_0^t \mid \partial_x S(t-s) f(x,s) \mid_{L^2((1+x)dx)} ds \mid \psi(t) \mid dt$$

$$\leq \int_0^T \int_s^T \mid \psi(t) \mid \mid \partial_x S(t-s) f(x,s) \mid_{L^2((1+x)dx)} dt ds$$

thanks to Fubini's theorem

$$\leq \int_0^T \left(\left(\int_0^T \mid \psi(t) \mid^2 dt \right)^{1/2} \left(\int_0^T \mid \partial_x S(t-s) f(x,s) \mid^2_{L^2((1+x)dx)} dt \right)^{1/2} ds \right.$$

thanks to Cauchy-Schwarz inequality

$$\leq c \left(\int_0^T \mid \psi(t) \mid^2 dt \right)^{1/2} \int_0^T \mid f(.,s) \mid_{L^2((1+x^2)dx)} ds \text{ thanks to (5)}$$

$$\leq c \left(\int_0^T \mid \psi(t) \mid^2 dt \right)^{1/2} \mid f \mid_{L^1(0,T;L^2((1+x^2)dx))} .$$

Inequality (16) therefore follows. By an analogous argument, we have the estimate (17). In order to obtain (18), (19), (20), (21) we multiply (14) by v_{xx}:

$$\frac{d}{dt} \int_0^L v_x^2(x,t) dx + v_x^2(0,t) + v_{xx}^2(0,t) = -2 \int_0^L f(x,t) v_{xx}(x,t) dx. \tag{25}$$

Multiplying (14) by $x v_{xx}$ gives

$$\frac{d}{dt} \int_0^L x v_x^2(x,t) dx - \int_0^L v_x^2(x,t) dx + 3 \int_0^L v_{xx}^2(x,t) dx + 2 v_x(0,t) v_{xx}(0,t)$$

$$= -2 \int_0^L v_x(x,t) f(x,t) dx - 2 \int_0^L x v_{xx}(x,t) f(x,t) dx. \tag{26}$$

The combination 2×(25)+(26) shows that

$$\begin{aligned} &\tfrac{d}{dt} \int_0^L (2+x) v_x^2(x,t) dx + v_x^2(0,t) + v_{xx}^2(0,t) + 2 \int_0^L v_{xx}^2(x,t) dx \\ &\leq 2 \int_0^L v_x^2(x,t) dx + c \int_0^L (1+x^2) f^2(x,t) dx. \end{aligned} \tag{27}$$

Multiplying (27) by s and integrate in the temporal variable over $(0,t)$ leads to the estimates (18), (19), (20), (21) thanks to the inequality (15).

We finally integrate (27) over $(0,t)$ and inequalities (22), (23), (24) now follow and the proposition 2 is proved. ■

2.1.3 Nonlinear case

We now prove in this section a local existence result for the nonlinear system (2).

Proposition 3 *Assume that $L \geq 1$ and $g(0) = u_0(0)$ then there exists a time $T > 0$ independent of L and an unique weak solution u of (2) on $[0,T]$.*

Proof : the proof is splitted in three steps.

1. First, we make a relevement of the boundary data as follows. Recall that ϕ is a smooth function over $\mathbb{R}^+$ such that

$$\phi(0) = 1, \ \phi(x) = 0, \ \forall x \geq 1$$

and let $\tilde{u}$ be defined by

$$\tilde{u}(x,t) = u(x,t) - \phi(x)g(t).$$

System (2) becomes

$$\begin{cases} \tilde{u}_t + \tilde{u}_x + \tilde{u}_{xxx} = -F(\tilde{u}, \tilde{u}_x, g), \ x \in [0,L], \ t \in [0,T[, \\ \tilde{u}(0,t) = 0, \ t \in [0,T[, \\ \tilde{u}_x(L,t) = 0, \ t \in [0,T[, \\ \tilde{u}_{xx}(L,t) = 0, \ t \in [0,T[, \\ \tilde{u}(x,0) = u_0(x) - \phi(x)g(0), \ x \in [0,L], \end{cases}$$

where

$$F(\tilde{u}, \tilde{u}_x, g) = (\tilde{u} + \phi g)(\tilde{u}_x + \phi' g) + g(\phi' + \phi''') + \phi g'.$$

As a result, we have transformed the original problem (2) into a problem with a Dirichlet boundary condition $g = 0$. In what follows, in order to simplify the notations, we omit the tilde and we still denote the unknow by u instead of $\tilde{u}$.

2. We write $u(x,t) = S(t)u_0(x) - \displaystyle\int_0^t S(t-s)F(u,u_x,g)(s)ds$ where $S(t)$ is the linear semi-group introduced in the section 2.1. We refer to section 2.1.1 for the part $S(t)u_0$ and to section 2.1.2 for the part $\int_0^t S(t-s)F(u,u_x,g)(s)ds$. We introduce the following functional $\mathcal{T}$ defined by

$$\mathcal{T}(u_0, g, u) := S(t)u_0(x) - \int_0^t S(t-s)F(u,u_x,g)(s)ds.$$

One has

Lemma 1 *There exists a constant $c(T)$ depending on T, independent of L such that for all $u_0 \in L^2(0,L)$:*

$$\mid S(t)u_0 \mid_X \leq c(T) \mid u_0 \mid$$

and the map $T \mapsto c(T)$ is continuous.

Proof : it is an obvious consequence of inequalities (4), (5), (9), (10). ■

For the non homogeneous problem the following lemma holds :

Lemma 2 *There exists a constant $c(T)$ depending on T, independent of L such that for all f in E*

$$\mid \int_0^t S(t-s)f(s)ds \mid_X \leq c(T) \mid f \mid_E$$

and the map $T \mapsto c(T)$ is continous.

Proof : it is an obvious consequence of inequalities (15),(16), (18), (21). ■

3. Contraction procedure : we begin with a lemma :

Lemma 3 *For all w in $H^1([0,L])$ such that $w(0)=0$, one has*

$$\mid \sqrt{x}w \mid_{L^\infty} \leq 5\left(\sqrt{\mid w \mid_{L^2(1+x)}}\sqrt{\mid w_x \mid_{L^2(1+x)}} + \mid w \mid_{L^2(1+x)}\right). \tag{28}$$

Proof : first, one has

$$\sup_{0\leq x\leq 1} \mid \sqrt{x}w \mid \leq \sup_{0\leq x\leq 1} \mid w \mid \leq \mid w \mid_{L^\infty} \leq \sqrt{2}\sqrt{\mid w \mid_{L^2}}\sqrt{\mid w_x \mid_{L^2}}$$

since $w(0,t)=0$.

Concerning $x \geq 1$, one has :

$$\sup_{x\geq 1} \mid \sqrt{x}w \mid \leq \mid w(1) \mid + \sqrt{2}\sqrt{\mid \sqrt{x}w \mid_{L^2}}\sqrt{\mid \frac{\partial}{\partial x}(\sqrt{x}w) \mid_{L^2}} \text{ since } w(0)=0$$

$$\leq \sqrt{2}\sqrt{\mid w \mid_{L^2}}\sqrt{\mid w_x \mid_{L^2}} + \sqrt{2}\sqrt{\mid \sqrt{x}w \mid_{L^2}}\sqrt{\mid w \mid_{L^2} + \mid \sqrt{x}w_x \mid_{L^2}} \text{ because } x \geq 1.$$

Finally,

$$\mid \sqrt{x}w \mid_{L^\infty} \leq 2\sqrt{2}\sqrt{\mid w \mid_{L^2}}\sqrt{\mid w_x \mid_{L^2}} + \sqrt{2}\sqrt{\mid \sqrt{x}w \mid_{L^2}}\sqrt{\mid w \mid_{L^2} + \sqrt{x}w_x \mid_{L^2}}$$

$$\leq 2\sqrt{2}\sqrt{\mid w \mid_{L^2(1+x)}}\sqrt{\mid w_x \mid_{L^2(1+x)}} + \sqrt{2}\sqrt{\mid w \mid_{L^2(1+x)}}\sqrt{\mid w \mid_{L^2(1+x)} + \mid w_x \mid_{L^2(1+x)}}$$

$$\leq 3\sqrt{2}\sqrt{\mid w \mid_{L^2(1+x)}}\sqrt{\mid w_x \mid_{L^2(1+x)}} + \sqrt{2} \mid w \mid_{L^2(1+x)}$$

$$\leq 5(\sqrt{\mid w \mid_{L^2(1+x)}}\sqrt{\mid w_x \mid_{L^2(1+x)}} + \mid w \mid_{L^2(1+x)}) \ (31).$$

■

The following estimate is the keypoint of this section :

Proposition 4 *Suppose that $u_0,\ v_0$ are in $L^2(0,L)$, g and h are in $H^1_{loc}(\mathbb{R}^+)$, then there exists a continuous function $t \mapsto c(t)$ such that for all T in $[0,T_0]$,*

$$\mid \mathcal{T}(u_0,g,u) - \mathcal{T}(v_0,h,v) \mid_X \leq$$

$$c(T) \mid u_0 - v_0 \mid + c(T)\sqrt{T}(\mid g \mid_{H^1(0,T)} + \mid h \mid_{H^1(0,T)} + 1 + \mid u \mid_X) \mid g - h \mid_{H^1(0,T)}$$
$$+ c(T)T^{1/4} \mid u - v \mid_X (\mid u \mid_X + \mid v \mid_X)$$
$$+ c(T)\sqrt{T} \mid u - v \mid_X (\mid h \mid_{H^1(0,T)} + \mid u \mid_X + \mid v \mid_X).$$

Proof :
We have

$$\mathcal{T}(u_0,g,u) - \mathcal{T}(v_0,h,v) = S(t)(u_0 - v_0)(x) - \int_0^t S(t-s)(F(u,u_x,g) - F(v,v_x,h))(s)ds.$$

We introduce the following quantities :

$$G(s) = F(u,u_x,g)(s) - F(v,v_x,h)(s) = F_c + F_l + F_{nl},$$

$$F_c = \phi\phi'(g^2 - h^2) + (g-h)(\phi' + \phi''') + \phi(g' - h'),$$
$$F_l = (u\phi' g - v\phi' h) + (\phi g u_x - \phi h v_x),$$
$$F_{nl} = uu_x - vv_x.$$

Thanks to the lemma 1 and 2, one gets

$$\mid \mathcal{T}(u_0,g,u) - \mathcal{T}(v_0,h,v) \mid_X \leq c(T) \mid u_0 - v_0 \mid$$

$$+ c(\mid G \mid_{L^1(0,T;L^2(1+x^2))} + \mid \sqrt{t}G \mid_{L^2(0,T;L^2(1+x^2))}).$$

Now, we want to control G in $L^1(0,T;L^2(1+x^2))$ and $\sqrt{t}G$ in $L^2(0,T;L^2(1+x^2))$. We estimate separately the three parts of G : $F_c,\ F_l$ and F_{nl}.

1. The term independent of u and v (i.e F_c):

On the one hand, one has

$$\mid F_c \mid_{L^2((1+x^2)dx)} \leq c(1 + \mid g(t) \mid + \mid h(t) \mid) \mid g(t) - h(t) \mid + c \mid g'(t) - h'(t) \mid$$

since ϕ is compactly supported. On the other hand, one gets

$$\mid F_c \mid_E \leq \sqrt{T} \mid F_c \mid_{L^2(0,T;L^2((1+x^2)dx)}$$

so that

$$\mid F_c \mid_E \leq c\sqrt{T}(1 + \mid h \mid_{H^1(0,T)} + \mid g \mid_{H^1(0,T)}) \mid g - h \mid_{H^1(0,T)}$$

since $H^1(0,T) \subset L^\infty(0,T)$.

2. The linear term F_l :

One has for the first term of F_l

$$\begin{aligned}
\mid u\phi' g - v\phi' h \mid_{L^2(0,T;L^2(1+x^2))} &\leq \mid \phi' u(g-h) \mid_{L^2(0,T;L^2(1+x^2))} \\
&\quad + \mid \phi'(u-v)h) \mid_{L^2(0,T;L^2(1+x^2))} \\
&\leq c \mid g-h \mid_{L^\infty(0,T)} \mid u \mid_{L^2(0,T;L^2(1+x))} \\
&\quad +c \mid h \mid_{L^\infty(0,T)} \mid u-v \mid_{L^2(0,T;L^2(1+x))} \\
&\quad \text{ since } \phi \text{ is compactly supported} \\
&\leq c \mid g-h \mid_{H^1(0,T)} \mid u \mid_X + c \mid h \mid_{H^1(0,T)} \mid u-v \mid_X \\
&\leq c(\mid u \mid_X \mid g-h \mid_{H^1(0,T)} + \mid u-v \mid_X \mid h \mid_{H^1(0,T)}),
\end{aligned}$$

and for the second term of F_l we have :

$$\begin{aligned}
\mid \phi(u_x g - v_x h) \mid_{L^2(0,T;L^2(1+x^2))} &\leq \mid \phi u_x(g-h) \mid_{L^2(0,T;L^2(1+x^2))} \\
&\quad + \mid \phi(u_x - v_x)h \mid_{L^2(0,T;L^2(1+x^2))} \\
&\leq c \mid g-h \mid_{L^\infty(0,T)} \mid u_x \mid_{L^2(0,T;L^2(1+x))} \\
&\quad +c \mid h \mid_{L^\infty(0,T)} \mid u_x - v_x \mid_{L^2(0,T;L^2(1+x))} \\
&\quad \text{ since } \phi \text{ is compactly supported} \\
&\leq c \mid g-h \mid_{H^1(0,T)} \mid u \mid_X + c \mid h \mid_{H^1(0,T)} \mid u-v \mid_X .
\end{aligned}$$

We therefore obtain

$$\mid F_l \mid_E \leq c\sqrt{T}(\mid u \mid_X \mid g-h \mid_{H^1(0,T)} + \mid u-v \mid_X \mid h \mid_{H^1(0,T)}).$$

3. The nonlinear term F_{nl} :
in order to estimate

$$\mid uu_x - vv_x \mid_{L^1(0,T;L^2((1+x^2)dx))} + \mid \sqrt{t}(uu_x - vv_x) \mid_{L^2(0,T;L^2((1+x^2)dx))},$$

we write $uu_x - vv_x = (u-v)u_x + v(u_x - v_x)$. For the first term, we want a bound of UV_x in $L^1(0,T;L^2((1+x^2)dx))$ where $U = u - v$ with $V = u$. First one has,

$$\begin{aligned}
\mid UV_x \mid_{L^1(0,T;L^2((1+x^2)dx))} &= \int_0^T \mid UV_x \mid_{L^2((1+x^2)dx)} dt \\
&= \int_0^T \sqrt{\int_0^L (1+x^2)U^2V_x^2 dx} dt \\
&= \int_0^T \sqrt{\int_0^L U^2V_x^2 dx + \int_0^L x^2U^2V_x^2 dx} dt \\
&\leq \sqrt{2}\int_0^T (\sqrt{\int_0^L U^2V_x^2 dx} + \sqrt{\int_0^L x^2U^2V_x^2 dx}) dt.
\end{aligned}$$

Moreover:

$$\int_0^L U^2V_x^2dx \leq \mid U \mid_{L^\infty}^2 \int_0^L (1+x)V_x^2 dx \leq 2 \mid U \mid_{L^2(1+x)} \mid U_x \mid_{L^2(1+x)} \mid V_x \mid_{L^2(1+x)}^2$$

and

$$\int_0^L x^2U^2V_x^2dx \leq \mid xU^2 \mid_{L^\infty} \int_0^L xV_x^2dx \leq \mid \sqrt{x}U \mid_{L^\infty}^2 \int_0^L (1+x)V_x^2dx.$$

Therefore

$$\begin{aligned} \mid xUV_x \mid_{L^2} &\leq \mid \sqrt{x}U \mid_{L^\infty} \sqrt{\textstyle\int_0^L (1+x)V_x^2dx} \\ &\leq 5(\sqrt{\mid U \mid_{L^2(1+x)}}\sqrt{\mid U_x \mid_{L^2(1+x)}} + \mid U \mid_{L^2(1+x)}) \mid V_x \mid_{L^2(1+x)} \\ &\quad \text{thanks to the lemma 2.} \end{aligned}$$

Hence,

$$\begin{aligned} \mid UV_x \mid_{L^1(0,T;L^2((1+x^2)dx)} \leq \sqrt{2} \int_0^T \mid U \mid_{L^2(1+x)}^{1/2} \mid U_x \mid_{L^2(1+x)}^{1/2} \mid V_x \mid_{L^2(1+x)} dt \\ +5\int_0^T \left(\mid U \mid_{L^2(1+x))}^{1/2} \mid U_x \mid_{L^2(1+x)}^{1/2} + \mid U \mid_{L^2(1+x)}\right) \mid V_x \mid_{L^2(1+x)} dt \\ \leq c\int_0^T \mid U \mid_{L^2(1+x)}^{1/2} \mid U_x \mid_{L^2(1+x)}^{1/2} \mid V_x \mid_{L^2(1+x)} dt \\ +5\int_0^T \mid U \mid_{L^2(1+x))} \mid V_x \mid_{L^2(1+x)} dt \\ \leq c \mid U \mid_X \int_0^T \mid U_x \mid_{L^2(1+x)}^{1/2} \mid V_x \mid_{L^2(1+x)} dt \\ +5 \mid U \mid_X \int_0^T \mid V_x \mid_{L^2(1+x)} dt \\ \leq c(\sqrt{T} + T^{1/4}) \mid U \mid_X \mid V \mid_X, . \end{aligned}$$

This last result yields

$$\mid (u-v)u_x \mid_{L^1(0,T;L^2((1+x^2)dx))} \leq c(\sqrt{T}+T^{1/4}) \mid u-v \mid_X \mid u \mid_X .$$

We write U instead of v, V instead of $u-v$ and we use the same technique for the second term:

$$\begin{aligned} \mid UV_x \mid_{L^1(0,T;L^2((1+x^2)dx))} = \mid v(u_x - v_x) \mid_{L^1(0,T;L^2((1+x^2)dx))} \\ \leq c(\sqrt{T}+T^{1/4}) \mid u-v \mid_X \mid v \mid_X . \end{aligned}$$

Hence, the estimate on $| uu_x - vv_x |_{L^1(0,T;L^2((1+x^2)dx))}$ is established. The second step is to estimate $| \sqrt{t}UV_x |_{L^2(0,T;L^2((1+x^2)dx))}$, where U is successively $u-v$ or v with $V=u$ or $V=u-v$. We write

$$| \sqrt{x}U\sqrt{x}V_x |_{L^2(dx)} \leq | \sqrt{x}U |_{L^\infty} | \sqrt{x}V_x |_{L^2(dx)}$$

$$\leq 5(\sqrt{| U |_{L^2(1+x)}}\sqrt{| U_x |_{L^2(1+x)}} + | U |_{L^2(1+x)}) | \sqrt{x}V_x |_{L^2(dx)}$$

thanks to (31)

$$\leq 5\sqrt{| U |_{L^2(1+x)}}\sqrt{| U_x |_{L^2(1+x)}} | V_x |_{L^2((1+x)dx)} + 5 | U |_{L^2(1+x)} | V_x |_{L^2((1+x)dx)}$$

Then

$$| \sqrt{t} | \sqrt{x}U\sqrt{x}V_x |_{L^2(dx)} |_{L^2(0,T)} \leq$$

$$5\sqrt{| U |_{L^\infty(0,T;L^2(1+x))}} | \sqrt{t}V_x |_{L^\infty(0,T;L^2(1+x))} \sqrt{| U_x |_{L^2(0,T;L^2((1+x)dx))}}\sqrt{T}$$

$$+5 | U |_{L^\infty(0,T;L^2(1+x))} | \sqrt{t}V_x |_{L^2(0,T;L^2((1+x)dx))}$$

$$\leq c | U |_X^{1/2} | V |_X \sqrt{| U_x |_{L^2(0,T;L^2((1+x)dx))}}\sqrt{T}$$

$$+c | U |_X \sqrt{T} | V |_X$$

$$\leq c\sqrt{T} | U |_X | V |_X, \ \forall 0 \leq t \leq T.$$

Finally $| \sqrt{t}UV_x |_{L^2(0,T;L^2(1+x^2))} \leq c\sqrt{T} | U |_X | V |_X$. Hence, the desired result follows and proposition 4 is established. ∎

Let $T_0 > 0$. Take $R = 2c(T_0)(| u_0 | + \sqrt{T_0}(1 + | g |_{H^1}) | g |_{H^1})$. We denote by B_R the ball of center 0 and of radius R in X_T.

Proposition 5 *Assume g in $H^1_{loc}(\mathbb{R}^+)$. Then, there exists a time T_1 in $]0,T_0]$ such that the application $u \mapsto \mathcal{T}(u_0,g,u)$ maps the ball B_R into itself.*

Proof : applying proposition 4 with $v_0 = 0,\ h = 0, v = 0$, one gets

$$| \mathcal{T}(u_0,g,u) |_X \leq c(T_0) | u_0 | +$$

$$c\sqrt{T}(| u |_X + | g |_{H^1(0,T)} + 1) | g |_{H^1(0,T)} + c | u |_X^2 T^{1/4} + c\sqrt{T} | u |_X^2 .$$

It follows

$$| \mathcal{T}(u_0,g,u) |_X \leq \frac{R}{2} + c\sqrt{T} | u |_X | g |_{H^1} + c | u |_X^2 T^{1/4} + c\sqrt{T} | u |_X^2 .$$

Then, if $u \in \mathcal{B}_R$ then

$$| \mathcal{T}(u) |_X \leq \frac{R}{2} + c\sqrt{T}R | g |_{H^1} + cT^{1/4}R^2 + c\sqrt{T}R^2.$$

Choosing T such that $c(T^{1/4}R^2 + \sqrt{T} | g |_{H^1(0,T_0)} R + \sqrt{T}R^2) \leq \frac{R}{2}$, ensures that $u \mapsto \mathcal{T}(u_0,g,u)$ maps the ball $\mathcal{B}_R$ into itself and proposition 5 is proved. ∎

Proposition 6 *Assume g in $H^1_{loc}(\mathbb{R}^+)$. Then, there exists a time T_2 in $]0, T_1]$ such that the application $u \mapsto \mathcal{T}(u_0, g, u)$ is a contraction over $(B_R, |\ .\ |_X)$.*

Proof : applying proposition 4 where $v_0 = u_0,\ h = g$ yields

$$| \mathcal{T}(u_0, g, u) - \mathcal{T}(u_0, g, v) |_X \leq cT^{1/4} | u - v |_X (| u |_X + | v |_X)$$

$$+c\sqrt{T} | u - v |_X (| g |_{H^1(0,T)} + | u |_X + | v |_X),$$

so that if $u,\ v$ in B_R then

$$| \mathcal{T}(u_0, g, u) - \mathcal{T}(u_0, g, v) |_X \leq c | u - v |_X (2RT^{1/4} + 2R\sqrt{T} + \sqrt{T} | g |_{H^1(0,T)}).$$

So that, if T is small enough namely

$$c(2RT^{1/4} + 2R\sqrt{T} + \sqrt{T} | g |_{H^1(0,T)}) < 1$$

then the application $u \mapsto \mathcal{T}(u_0, g, u)$ is a contraction over $(B_R, |\ .\ |_X)$. ∎

Proposition 7 *There exists a unique u defined in X_T, weak solution of (2).*

Proof : to prove this proposition, it is enough to apply the Banach's fixed point theorem for $u \mapsto \mathcal{T}(u_0, g, u)$ on B_R (which is a complete metric space) which yields local existence and uniqueness. As usual, one can then speak of maximal solutions. ∎

2.1.4 Continuous dependance

Proposition 8 *The solution u depends continuously on u_0 in $L^2((1 + x^2)dx)$ and g in $H^1_{loc}(\mathbb{R}^+)$.*

Proof : once again, this follows from Proposition 4 for small times. Namely, applying proposition 4, one gets :

$$| u - v |_X \leq c(T_0) | u_0 - v_0 | + c\sqrt{T}(| g |_{H^1} + | h |_{H^1} + 1 + | u |_X) | g - h |_{H^1(0,T)}$$

$$+c\sqrt{T} | u - v |_X (| h |_{H^1} + | u |_X + | v |_X) + cT^{1/4} | u - v |_X (| u |_X + | v |_X)).$$

Then if u_0 tends towards v_0 in $L^2(1 + x^2)$ and if g tends towards h in H^1 one gets that u tends towards v in X_T. These results were obtained locally in time. But, since the time interval where this result holds depends only on $| u_0 |$ and $| g |_{H^1}$, it can be extended as long the solution exists, which ends the proof of Theorem 3. ∎

2.2 Global existence and uniqueness for the quarter-plane problem ($L = +\infty$)

We introduce the space :

$$\tilde{X}_T := \{w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^+, (1+x^2)dx)), \text{ such that } w_x \in L^2(0,T; L^2(\mathbb{R}^+, (1+x)dx)),$$
$$\sqrt{t}w_x \in L^\infty(0,T; L^2(\mathbb{R}^+, (1+x)dx)), \ \sqrt{t}w_{xx} \in L^2(0,T; L^2(\mathbb{R}^+, dx))\}.$$

Theorem 4 *Let u_0 in $L^2(\mathbb{R}^+, (1+x^2)dx)$, g in $H^1_{loc}(\mathbb{R}^+)$. Then, there exists a unique u in $\mathcal{C}(\mathbb{R}^+; L^2(\mathbb{R}^+, (1+x^2)dx))$ solution of (1) such that u belongs to $\tilde{X}_T$ for all $T > 0$. Moreover, for all $t > 0$, u belongs to $\mathcal{C}([t, +\infty[; H^2)$ and for all $t > 0$, $u(x,t)$ is the solution obtained by Bona and Winther.*

Proof : All the estimates obtained in the previous section apply to the quarter plane problem ($L = +\infty$) since they are uniform with respect to L. This gives local existence and uniqueness in the space $\tilde{X}_T$. In order to prove that the solution is global, we need to establish some energy estimate. For the sake of simplicity, we compute them in the case $g \equiv 0$. The general case can be handle as in [3] using the change of function $v = u - g(t)e^{-x}$.
Multiplying the equation by u yields

$$\frac{d}{dt}\int_0^{+\infty} | u |^2 \, dx + u_x^2(0,t) = 0$$

namely $| u |_{L^2} \leq | u_0 |_{L^2}$. Multiplying the equation by xu gives

$$\frac{d}{dt}\int_0^{+\infty} x | u |^2 \, dx - \int_0^{+\infty} | u |^2 \, dx - \frac{2}{3}\int_0^{+\infty} | u |^3 \, dx + 3\int_0^{+\infty} u_x^2 dx = 0$$

so that

$$\frac{d}{dt}\int_0^{+\infty} x | u |^2 \, dx + 3\int_0^{+\infty} u_x^2 dx \leq c + c | u_x |_{L^2}^{1/2} .$$

And therefore

$$\frac{d}{dt}\int_0^{+\infty} x | u |^2 \, dx \leq c,$$

and $\int_0^T \int_0^{+\infty} u_x^2 dxdt \leq cT$. Multiplying by $x^2 u$ leads to

$$\frac{d}{dt}\int_0^{+\infty} x^2 | u |^2 \, dx - 2\int_0^{+\infty} x | u |^2 \, dx - \frac{4}{3}\int_0^{+\infty} x | u |^3 \, dx + 6\int_0^{+\infty} x | u_x |^2 \, dx = 0,$$

which implies

$$\frac{d}{dt}\int_0^{+\infty} x^2 | u |^2 \, dx + 6\int_0^{+\infty} x u_x^2 dx \leq c + c | u_x |_{L^2}^2 .$$

After integration in time, one obtains

$$\int_0^{+\infty} x^2 | u |^2 \, dx + 6\int_0^T \int_0^{+\infty} x | u_x |^2 \, dxdt \leq c + \int_0^T \int_0^{+\infty} u_x^2 dxdt.$$

We therefore obtain global existence in the quarter plane. Since $u(.,t)$ in H^2 for a.e. t, it is the solution of Bona and Winther.

- Now, we prove the uniqueness for the solution of (1) : we know that the solutions u belong to $L^\infty(0,T;L^2)$, that u_x belong to $L^2(0,T;L^2)$, and that $\sqrt{t}u_{xx}$ belong to $L^2(0,T;L^2)$. Let u and v be two solutions and we introduce : $w = u - v$. This function satisfies $w_t + w_x + w_{xxx} + wu_x + vw_x = 0$. Multiply by w and integrate the resulting expression with respect to the space variable x on $[0,+\infty[$, there appears :

$$\frac{d}{dt}\int_0^{+\infty} w^2(x,t)dx + w_x^2(0,t) + 2\int_0^{+\infty} w^2(x,t)u_x(x,t)dx - \int_0^{+\infty} w^2(x,t)v_x(x,t)dx,$$

namely

$$| u_x |_{L^\infty} \leq \sqrt{| u_x |_{L^2}}\sqrt{| u_{xx} |_{L^2}} = (\frac{1}{t^{\frac{1}{4}}}\sqrt{| u_x |_{L^2}})(t^{\frac{1}{4}}\sqrt{| u_{xx} |_{L^2}}).$$

But, we have $| u_x |_{L^2}^{\frac{1}{2}} \in L_t^4$, $t^{\frac{1}{4}}\sqrt{| u_{xx} |_{L^2}} \in L_t^4$, $t^{\frac{-1}{4}} \in L_t^{4-\varepsilon}$ then $| u_x |_{L^\infty} \in L_t^1$, $\forall \varepsilon \in]0,1[$ since $\frac{1}{4-\varepsilon} + \frac{1}{4} + \frac{1}{4} < 1$.

∎

3 Convergence towards the solution of the quarter plane problem for the homogeneous case ($g = 0$) when L tends towards $+\infty$

For the sake of simplicity, we restrict ourselves to the case $g = 0$. The result is of course valid if we suppose that g belongs to $H^1(\mathbb{R}^+)$. The aim of this section is to prove the following result:

Theorem 5 *Consider a family of initial values $u_0^L \in L^2([0,L],(1+x^2)dx)$ such that*

$$\sup_{L\geq 1}\int_0^L |u_0^L|^2(x)(1+x^2)dx < +\infty$$

and such that u_0^L tends towards u_0 in $L^2_{loc}(\mathbb{R}^+)$ strongly.
Then, for all $T > 0$, if L is large enough, $u^L(x,t)$ the solution of (2) with initial value u_0^L is defined on $[0,T]$ and u^L tends towards u in $L^p(0,T;L^2_{loc}(\mathbb{R}^+))$ strongly for all $p < +\infty$, where $u(x,t)$ is a solution of (1) with initial data u_0.

In order to prove this theorem, we will perform some energy estimates on the nonlinear equation. We therefore need more regular solutions:

Theorem 6 *Suppose u_0 in $H^3(0,L)$ and $\frac{\partial^3 u_0}{\partial x^3}$ in $L^2((1+x^2)dx)$. Suppose g in $H^2_{loc}(\mathbb{R}^+)$. Then u, the solution of (2) given by Theorem 3 satisfies $u_{xxx} \in X_T$, $\forall T < T_L$ where T_L is the maximum existence time of u.*

Proof : one has to solve the integral equation $\mathcal{T}(u_0, g, u) = u$ in the space $Y_T = \{u \in X_T,\ u_{xxx} \in X_T\}$. One finds a local in time solution, just as in section 2. Once this local solution in Y_T is constructed, one shows that its existence time as a solution in Y_T is the same as that as a solution in X_T by standard means. We omit the details. ∎

3.1 Behavior of the existence time.

We prove :

Proposition 9 *One has* $\lim_{L \to +\infty} T_L = +\infty$.

In order to prove this result, we will use some estimates of u in $L^\infty(0, T; H^1)$. Since the initial value is not in H^1, this is clearly not possible. However, for allmost avery $t > 0$, the solution $u(\cdot, t)$ lies in H^1. We therefore consider the initial value problem at some new origine of time t_L such that $u(\cdot, t_L) \in H^1$. An uniform control (with respect to L) of $|u(\cdot, t_L)|_{H^1}$ is given in the following lemma:

Lemma 4 *For all $L > 0$ there exists a time t_L such that*

$$\int_0^L (1+x) u_x^2(x, t_L) dx \leq \frac{4c}{T_2},$$

where T_2 is the existence time given in section 2. This time T_2 is independent of L.

This lemma is easily proven by contradiction using the fact that

$$\int_0^{T_2} \int_0^L (1+x) u_x^2(x,t) dx dt$$

is bounded independently of L. ∎

In order to show that T_L tends toward $+\infty$, we will consider problem (2) with initial data $u(\cdot, t_L)$. For the sake of simplicity, we still note $t = 0$ and $u_0 = u(\cdot, t_L)$.

We now adapt the method used in [9] for global existence for small data and we introduce the time dependent function X :

$$X(t) = \sup\left(\int_0^t \mid (1+\sqrt{x}) u \mid_{L^\infty}^4 (s) ds, 1\right).$$

We remark that $X(t)$ is nondecreasing and $X(t) \geq 1$.

We also introduce the fonction Y defined by

$$Y(t) = \int_0^L (1+x)(u_x^2 - u^2 - u^3/3) dx.$$

The idea of the proof is to show that $X(t)$ controls the norm $|u|$ of the solution. Then one proves that $X(t)$ also controls $Y(t)$ which itself controls $|(1+\sqrt{(x)})u_x|_{L^2}$. We therefore obtain an inequality for $X(t)$ in which the coefficients of the nonlinear terms are always proportionnal to some positive power of $1/L$.

Estimates on X:

We consider the IBVP (2) with $g=0$. Multiplying by u gives

$$\frac{d}{dt}\int_0^L u^2(x,t)dx + u^2(L,t) + \frac{2}{3}u^3(L,t) + u_x^2(0,t) = 0.$$

Integrate in the temporal variable leads to

$$\int_0^L u^2(x,t)dx - \int_0^L u_0^2(x)dx + \int_0^t u_x^2(0,s)ds \leq \frac{1}{L^{3/2}}\int_0^t |\,(1+\sqrt{x})u\,|_{L^\infty}^3 (s)ds.$$

Thanks to the Cauchy-Schwarz inequality, one gets

$$\int_0^L u^2(x,t)dx + \int_0^t u_x^2(0,s)ds \leq c_0 + \frac{t^{1/4}}{L^{3/2}}X(t)^{3/4}, \tag{29}$$

where $c_0 = \displaystyle\int_0^L u_0^2(x)dx$. Multiplying by xu leads to

$$\frac{d}{dt}\int_0^L xu^2dx - \int_0^L u^2dx + Lu^2(L,t) + \frac{2}{3}Lu^3(L,t) - \frac{2}{3}\int_0^L u^3dx + 3\int_0^L u_x^2dx = 0.$$

We control $\displaystyle\int_0^L u^3dx$ by

$$|u|_{L^2}|u|_{L^\infty} \leq C|u|_{L^2}^{5/2}|u_x|_{L^2}^{1/2} \leq |u_x|_{L^2}^2 + C|u|_{L^2}^{10}$$

in order to obtain with (29)

$$\frac{d}{dt}\int_0^L xu^2dx + 2\int_0^L u_x^2dx \leq C + C\frac{t^{1/2}}{L^{3/2}}X^{5/3} - \frac{2}{3}Lu^3(L,t).$$

Integrate this expression over $(0,t)$ and since the function $X(t)$ is non decreasing, one gets

$$\int_0^L xu^2(x,t)dx + 2\int_0^t\int_0^L u_x^2(x,s)dxds \leq c + ct + \frac{ct^{1/2}}{\sqrt{L}}X^{5/3}(t). \tag{30}$$

Multiplying the equation by the flux $u+u^2/2+u_{xx}$ gives

$$\frac{d}{dt}\int_0^L (u_x^2 - u^2 - u^3/3)dx + u_{xx}^2(0,t) \leq 2u^2(L,t) + u^4(L,t).$$

Multiplying the equation by $x(u+u^2/2+u_{xx})$ leads to

$$\frac{d}{dt}\int_0^L x(u_x^2 - u^2 - u^3/3)dx \leq (u+\frac{1}{2}u^2)^2(L,t).$$

Therefore

$$Y'(t) \leq 2(u + \frac{1}{2}u^2)^2(L,t)$$

and by integration in time, it follows

$$Y(t) \leq 4\frac{t^{1/2}}{L}X(t)^{1/2} + \frac{2}{L^2}X(t).$$

Plugging in this last expression the value of $Y(t)$ gives

$$\int_0^L (1+x)u_x^2(x,t)dx \leq C + C\frac{t^{1/2}}{L}X(t) + \int_0^L (1+x)(u^2 + \frac{u^3}{3})(x,t)dx.$$

Using (29) and (30) leads to

$$\int_0^L (1+x)u_x^2(x,t)dx \leq C + C\frac{t^{1/2}}{L}X(t) + Ct + C\frac{t^{1/2}}{L^{1/2}}X^{5/3} + |u_x|_{L^2}^{1/2}|(1+\sqrt{x})u|_{L^2}^{5/2}.$$

Using Young's inequality in this last expression yields:

$$\int_0^L (1+x)u_x^2(x,t)dx \leq C + C\left(t + \frac{t^{1/2}}{\sqrt{L}}X^{5/3}\right)^{10/3}. \tag{31}$$

On the other hand, the definition of X implies (using (29), (30) and (31))

$$X'(t) \leq | (1+\sqrt{x})u |_{L^\infty}^4 \leq C\left(1 + t + \frac{t^{1/2}}{\sqrt{L}}X^{5/3}\right)^{13/3}.$$

Integrating this inequality in the temporal variable yields :

$$X(t) \leq 1 + C\left(1 + t + \frac{t^{1/2}}{\sqrt{L}}X^{5/3}\right)^{13/3},$$

that is

$$X(t) \leq C\left(1 + t^6 + \frac{t^4}{L^2}X^8\right). \tag{32}$$

We are now able to prove proposition 9.

Note first that function $X(t)$ controls the norm $|u|$ of the solution. Therefore, in order to conclude, we only need to prove that $X(t)$ is bounded on a time interval which length tends to infinity as $L \to +\infty$.

Take $T > 0$ and let $R = 2C(1+T)^6$. It follows from $X(0) = 1$ by continuation the existence of T' in $]0,T]$ such that $\sup_{t\in[0,T']} X(t) \leq R$, hence (32) implies

$$\sup_{t\in[0,T']} X(t) \leq \frac{R}{2} + CT^4R^8/L^2.$$

Choosing T such that $CT^4R^7/L^2 \leq 1/4$ leads to $\sup_{t\in[0,T']} X(t) \leq 3R/4$. This condition is fullfilled as soon as $T^4 2^7 C^7 (1+T^6)^7/L^2 \leq 1/4$ which is satisfied if $T \sim L^\alpha$ for some $\alpha > 0$. By continuation, it follows $T' = T$. Hence, the existence time T_L tends towards $+\infty$ when L tends towards $+\infty$. ■

3.2 End of proof of Theorem 5

Proof : let us take $T > 0$. Thanks to the preceeding proposition, if L is large enough, u^L is defined at least on $[0, T]$ and is bounded in $L^2(0, T; H^1_{loc})$. Moreover $\partial_t u^L$ is bounded in $L^2(0, T; H^{-2}_{loc})$. Therefore standard compactness argument yields convergence in $\mathcal{C}([0, T]; H^{-s}) \cap L^p(0, T; L^2_{loc})$ strongly for all $s > 0$ and $p < +\infty$ and in $L^2(0, T; H^1)$ weakly towards some function $u(x, t)$ lying in the same space. Therefore, $u(0, t)$ and $u(x, 0)$ make sense and are equal respectively to $g(t)$ and $u_0(x)$. Thanks to the strong compactness, it is straightforward that u satisfies the Korteweg-de Vries equation. Therefore, u is solution to the quaterplane problem. Since the limit is unique, all the sequence u^L converges towards u. ∎

4 Numerical approximation.

The aim of this section is to present a very simple finite-difference scheme for problem (2) on the space interval $[0, 1]$.

4.1 Description of the scheme and proof of its stability.

We take a time-step δt and a space-step δx. We denote by y_i^n the approximate value of the solution at time $n\delta t$ and at the point $i\delta x$.
We denote by X_N the following space of finite sequences

$$X_N = \{y = (y_0, y_1, \cdots, y_N) \in \mathbb{R}^{N+1},\ \text{with } y_0 = 0 \text{ and } y_N = y_{N-1} = y_{N-2}\}$$

endowed with the following inner product

$$\forall (y, z) \in X_N^2,\ (y, z) = \delta x \sum_{i=1}^{N-2} y_i z_i$$

and by the associated norm

$$|y| = (y, y)^{1/2}.$$

We also introduce the classical difference operators:

$$(D^+ y)_i = \frac{y_{i+1} - y_i}{\delta x} \text{ and } (D^- y)_i = \frac{y_i - y_{i-1}}{\delta x}$$

We first consider the linear problem asociated to (2):

$$\begin{cases} u_t + u_{xxx} = 0,\ x \in [0, 1],\ t \geq 0, \\ u(0, t) = u_x(1, t) = u_{xx}(1, t) = 0 \text{ for } t \geq 0, \\ u(x, 0) = u_0(x),\ \text{for } x \in [0, 1] \end{cases} \tag{33}$$

The scheme associated to (33) reads:

$$(S)\ \frac{y^{n+1}-y^n}{\delta t}+D^+D^+D^-y^{n+1}=0,$$

that is :

$$\frac{y_i^{n+1}-y_i^n}{\delta t}+\left\{\begin{array}{c}\dfrac{3y_1^{n+1}-3y_2^{n+1}+y_3^{n+1}}{\delta x^3}\ \text{for } i=1\\[2ex] \dfrac{y_{i+2}^{n+1}-3y_{i+1}^{n+1}+3y_i^{n+1}-y_{i-1}^{n+1}}{\delta x^3}\ \text{for } i=2,\cdots,N-4\\[2ex] \dfrac{-2y_{N-2}^{n+1}+3y_{N-3}^{n+1}-y_{N-4}^{n+1}}{\delta x^3},\ \text{for } i=N-3\\[2ex] \dfrac{y_{N-2}^{n+1}-y_{N-3}^{n+1}}{\delta x^3},\ \text{for } i=N-2\end{array}\right\}=0.$$

One first has

Proposition 10 *For all $y\in X_N$, one has:*

$$(D^+D^+D^-y,y)=\frac{1}{2}(D^-y)_1^2+\frac{\delta x}{2}|D^+D^-y|^2.$$

Before to prove this result, we remark that

Corollary 1 *For any $y^n\in X_N$ there exists an unique y^{n+1} satisfying (S).*

Proof: It is straightforward from proposition 10, since the matrix $I+\delta t D^+D^+D^-$ is clearly definite, positive.
We return now to the proof of proposition 10. We first prove:

Lemma 5 *For all sequences z and y such that $z_{N-1}=0$ and $y_0=0$, one has*

$$(D^+z,y)=-(z,D^-y).$$

This formula is well-known, we omit the proof. ■

It follows that, applying lemma 5 with $z=D^+D^-y$:

$$\forall y\in X_N,\ (D^+D^+D^-y,y)=-(D^+D^-y,D^-y). \tag{34}$$

In order to conclude, we need:

Lemma 6 *For any sequence z, one has*

$$(D^+z,z)=\frac{1}{2}z_{N-1}^2-\frac{1}{2}z_1^2-\frac{\delta x}{2}|D^+z|^2.$$

Proof:

$$\frac{1}{\delta x}(D^+z, z) = \sum_{i=1}^{N-2} \frac{z_{i+1} - z_i}{\delta x} z_i$$

$$= -\frac{1}{2} \sum_{i=1}^{N-2} \frac{z_i^2 - z_{i+1}^2}{\delta x} - \frac{1}{2\delta x}(z_{i+1} - z_i)^2,$$

where we have used the inequality

$$(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2. \tag{35}$$

The lemma follows. ■

Proposition 10 is then obtained thanks to (34) and lemma 6. ■

The following estimate shows that the scheme (S) is l^2-stable

Proposition 11 *For any y^n satisfying (S), one has*

$$|y^n|^2 + \delta t \sum_{k=0}^{n} \delta t \left| \frac{y^{k+1} - y^k}{\delta t} \right|^2$$

$$+ \sum_{k=0}^{n} \delta t |(D^- y^k)_1|^2 + \delta x \sum_{k=0}^{n} \delta t |D^+ D^- y^k|^2 = |y^0|^2.$$

Proof: Take the inner product of (S) with y^{n+1}, apply proposition 33 and equality 35. ■

In order to obtain the unconditionnal stability for the nonlinear version of the scheme, we will find a discrete estimates that is equivalent to that of u_x in $L^2(0,T;L^2)$ in the continuous case. Let us denote by x the sequence $x_i = i\delta x$. One has:

Proposition 12 *For all y^n satisfying (S), one has*

$$|y^n|^2 + \delta t \sum_{k=0}^{n} \delta t \left| \sqrt{x} \left(\frac{y^{k+1} - y^k}{\delta t} \right) \right|^2$$

$$+ \delta t \sum_{k=0}^{n} \left(3|z^k|^2 + \delta x \sum_{i=2}^{N-3} \delta x (i-1) \delta x |(D^+ z^k)_i|^2 + \right.$$

$$\left. \delta x (N-2)(z_{N-1}^k - z_{N-2}^k)^2 + \delta x (z_2^k - z_1^k)^2 \right) = |y^0|^2,$$

where $z^k = D^- y^k$.

Proof: As in the continuous case, one takes the inner product of (S) with xy^{n+1}. We omit the details. ■

We therefore have obtained

Proposition 13 *There exists a constant $C > 0$ (independent of δx and δt) such that any solution $(y^n)_{n=0,\cdots,p} \in X_N^{p+1}$ to*

$$\frac{y^{n+1} - y^n}{\delta t} + D^+ D^+ D^- y^{n+1} = 0$$

satisfies

$$\sup_{k=0,\cdots,p} |y^k|^2 + \delta t \sum_{k=0}^{p} |y^k|_1^2 \leq C|y_0|^2.$$

where $|y|_1 = |D^+ y|$.

Since X_N endowed with the norm $|y|_X = |y| + |y|_1$ is an algebra, *i.e.* there exists C independent of δx such that

$$\forall y, z \in X_N, \ |yz|_X \leq C|y|_X |z|_X$$

the existence proof of the continuous case of [9] applies in the discret nonlinear case for any discretisation of the nonlinear part, for example $f_n = \frac{1}{2} D^-(y^n)^2$ or $y^n D^- y^n$. One gets

Theorem 7 *Let $y^{0,\delta x} \in X_N$ be such that $\limsup_{\delta x \to 0} |y^{0,\delta x}| < +\infty$. There exists $\varepsilon_0 > 0$ such that if $\delta t \leq \varepsilon_0$, there exists $T > 0$ and a unique solution $(y^{n,\delta x,\delta t})_{n=0,\cdots,[T/\delta t]}$ to*

$$\frac{y^{n+1} - y^n}{\delta t} + D^+ D^+ D^- y^{n+1} = f^n$$

Moreover, there exits a constant C independent of δt and δx such that

$$\sup_{k=0,\cdots,p} |y^{k,\delta x,\delta t}|^2 + \ \delta t \sum_{k=0}^{p} |y^{k,\delta t,\delta x}|_1^2 \leq C|y_0|^2.$$

This result means that independently of the discretization of the nonlinear term, the scheme is unconditionnally stable.

4.2 Some numerical results.

We have implemented the scheme (S) using Scilab. The discretization used for the nonlinear term is explicit and is $y^n D^+ y^n$.
Performing a change of variables, we have solved the initial boundary value problem on $[0, L]$ with $L = 10$:

$$u_t + \frac{1}{L} u u_x + \frac{1}{L^3} u_{xxx} = 0.$$

We have taken $\delta t = 2,5.10^{-5}$, $\delta x = 5.10^{-5}$. We compute the solution during 4800 iterations in time, that is on the time interval $[0, T]$ with $T = 0.12$. The initial value is

$$u_0(x) = \frac{\alpha}{\cosh^2(\beta L(x - 1/2))} + \frac{4\alpha}{\cosh^2(4\beta L(x - 1/4))}.$$

with $\alpha = 12\beta^2$ and $\beta = 2$. This correspond to the superposition of two solitons with different speeds. The biggest one (which is also the fastest) is at the begining behind the smallest. In fig.1 we have represented the solution at time $t_i = iT/8$ for $i = 0, \cdots, 8$. The result is correct since one obtains the nonlinear interaction. However, it is less precise than the results obtained in [6] (or [1] for systems) in the periodic framework with higher order schemes. Here, our scheme is obviously of order one in time and space.

Figure 1: Interaction of two solitons.

These simultations show that this kind of boundary conditions can be used in order to compute solutions to the KdV equation without beeing in the periodic framework.

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